# Linear Independence and Wronskians : April 28 (and 30)

Suppose $f_1, \ldots, f_n$ are linearly dependent, i.e. there are constants $\underbrace{C_1, \ldots, C_n}_{\text{not all} = 0}$ with $C_1 f_1 + C_2 f_2 + \cdots + C_n f_n \equiv 0$ ($\equiv 0$ everywhere). Then differentiating this gives

$$C_1 f_1 + \cdots + C_n f_n \equiv 0$$
$$C_1 f_1' + \cdots + C_n f_n' \equiv 0$$
$$\vdots$$
$$C_1 f_1^{(n-1)} + \cdots + C_n f_n^{(n-1)} \equiv 0$$
$$\vdots$$

Choose an arbitrary point $x_0$. Then

$$C_1 f_1(x_0) + \cdots + C_n f_n(x_0) = 0$$
$$C_1 f_1'(x_0) + \cdots + C_n f_n'(x_0) = 0$$
$$\vdots \qquad\qquad \vdots$$
$$C_1 f_1^{(n-1)}(x_0) + \cdots + C_n f_n^{(n-1)}(x_0) = 0$$

Think of this as a system of equations, homogeneous, n equations, n "unknowns" $C_1, \ldots C_n$, coefficient matrix

$$\begin{pmatrix} f_1(x_0) & \cdots & f_n(x_0) \\ \vdots & & \\ f_1^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \end{pmatrix}$$

We already know that this system has a "nontrivial" solution, since we assumed not all the original C's were 0. So linear algebra implies

$$\det\begin{pmatrix} f_1(x_0) & \cdots & f_n(x_0) \\ \vdots & & \\ f_1^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \end{pmatrix} = 0.$$

So:

$f_1, \ldots f_n$ linearly dependent $\Rightarrow \det\begin{pmatrix} f_1(x_0) & \cdots & f_n(x_0) \\ & & \\ f_1^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \end{pmatrix} = 0.$

This works for every $x_0$. So

$$\det\begin{pmatrix} f_1 & \cdots & f_n \\ \vdots & & \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix} \equiv 0$$

Notation and terminology: this is called the Wronkian of $f_1, \ldots, f_n$ at $x_0$ denoted

$$W(f_1 \ldots f_n)\Big|_{x_0}.$$

Example: $f_1 = xe^x \quad f_2 = e^{2x} \quad f_3 = xe^x - e^{2x}$

dependent set $\qquad -f_1 + f_2 + f_3 \equiv 0$

$f_1' = e^x + xe^x \qquad f_2' = 2e^{2x} \qquad f_3' = e^x + xe^x - 2e^{2x}$

$f_1'' = 2e^x + xe^x \qquad f_2'' = 4e^{2x} \qquad f_3'' = 2e^x + xe^x - 4e^{2x}$

$$W(f_1,f_2,f_3)\big|_x = \det\begin{pmatrix} xe^x & e^{2x} & xe^x - e^{2x} \\ e^x + xe^x & 2e^{2x} & e^x + xe^x - 2e^{2x} \\ 2e^x + xe^x & 4e^{2x} & 2e^x + xe^x - 4e^{2x} \end{pmatrix}$$

This $\equiv 0$ because the third column = 1st column − 2nd column

"Independent" example: $e^{2x}, e^{3x}$

$$W(e^{2x}, e^{3x})\big|_x = \det\begin{pmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{pmatrix} = e^{5x}$$

W need not vanish anywhere!

Wronskian can vanish some places but not others:

Ex: $f_1 = \sin x \quad f_2 = \sin 5x \qquad W(f_1, f_2)\big|_x = \det\begin{pmatrix} \sin x & \sin 5x \\ \cos x & 5\cos 5x \end{pmatrix}$

so $W(f_1, f_2)\big|_x = 5 \sin x \cos 5x - \cos x \sin 5x$

This $= 0$ when (e.g.) $x = 0$. But when $x = \frac{\pi}{4}$,

$$W(f_1, f_2)\big|_{\pi/4} = 5\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -4\left(\frac{1}{2}\right) = -2 \neq 0.$$

This is consistent with the result on $W \equiv 0$ if functions are dependent, since $\sin x$ and $\sin 5x$ are independent so there is no reason for $W = 0$ everywhere but also no reason why $W$ cannot be 0 somewhere.

Natural question: If $W(f_1, \ldots, f_n) \equiv 0$, $W(f_1, f_2 \ldots f_n)\big|_x = 0$ all $x$, is it necessarily true that $f_1, \ldots f_n$ are dependent?

Answer: No. $W \equiv 0$ can happen even when $f_1, \ldots f_n$ are independent. But the examples are strange (functions not given by simple formulas).

HOWEVER, there is a statement sort of like this that is true:

If $y_1, \ldots, y_n$ are solutions of a (single) nth order linear homogeneous differential equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0 \qquad \text{(p's can be functions!)}$$

and if, for one $x_0$,

$$W(y_1, \ldots, y_n)\big|_{x_0} = 0,$$

then it must be that $y_1, \ldots, y_n$ are dependent

and hence $W(y_1, \ldots y_n)\big|_x = 0$ all $x$

i.e. $W(y_1, \ldots y_n) \equiv 0$.

How can this be consistent with the $\sin x$, $\sin 5x$ example? Answer: $\sin x$ and $\sin 5x$ do not solve the same 2nd order linear homogeneous differential equation ( $\sin x$ solves $y'' + y = 0$ while $y'' + 25y$ is the equation $\sin 5x$ solves, and therefore there is no 2nd order linear homog. equation they both solve: easy to check*).

Now we just need to see why $W(y_1, \ldots, y_n)|_{x_0} = 0$
$\Rightarrow$ $y_1 \ldots y_n$ are dependent (when $y$'s solve same equation):

Notice first that by linear algebra, $W(y_1, \ldots, y_n)|_{x_0} = 0$
$\Rightarrow$ there are constants $C_1, \ldots, C_n$ not all $= 0$ such that

$$
\begin{aligned}
C_1 y_1(x_0) + \cdots + C_n y_n(x_0) &= 0 \\
C_1 y_1'(x_0) + \cdots + C_n y_n'(x_0) &= 0 \\
&\vdots \\
C_1 y_1^{(n-1)}(x_0) + \cdots + C_n y_n^{(n-1)}(x_0) &= 0
\end{aligned}
$$

*The equation could not involve $y'$ since $(\sin x)'' + a_1(\sin x)' + a_2 \sin x = -\sin x + a_1 \cos x + a_2 \sin x$ can this cannot be $\equiv 0$ (we know why!) if $a_1 \neq 0$.

This is just like before: $\det = 0 \iff$ system has "nontrivial" solution when one thinks of $C_1, \ldots, C_n$ as unknowns.

Set $y = C_1 y_1 + \cdots + C_n y_n$. Then $y$ solves the differential equation since it is linear, homogeneous.

But

$$
\begin{aligned}
y(x_0) &= C_1 y_1(x_0) + \cdots + C_n y_n(x_0) = 0 \\
y'(x_0) &= C_1 y_1'(x_0) + \cdots + C_n y_n'(x_0) = 0 \\
&\vdots \\
y^{(n-1)}(x_0) &= C_1 y_1^{(n-1)}(x_0) + \cdots + C_n y_n^{(n-1)}(x_0) = 0.
\end{aligned}
$$

So the Uniqueness part of "existence & uniqueness"

$\Longrightarrow$ $y \equiv 0$ or $C_1 y_1 + \cdots + C_n y_n \equiv 0$

So $y_1, \ldots, y_n$ are dependent. $\square$

Notice that it follows logically that if $y_1, \cdots y_n$ are solutions of a linear $n$th order homogeneous equation, then either

(a) $W(y_1, \ldots y_n) \equiv 0$

or

(b) $W(y_1, \ldots y_n)\big|_{x_0}$ is never $0$

(but not both (a) and (b) of course!)

It is interesting that one can decide this "either/or" statement is true <u>without</u> going through any thoughts about dependence and independence. In particular:

$W = W(y_1, \dots y_n)$ satisfies the equation

$$W' = -p_1(x)\, W$$

so

$$W = C e^{-\int p_1}$$

Hence $W$ is either $\equiv 0$ (if $C=0$)

or never $0$ (if $C \neq 0$).

Why does $W$ satisfy this equation $W' = -p_1 W$?

To see why, we have to remember that determinants get differentiated "row by row" namely

$$W' = \det\begin{pmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & \cdots & & y_n^{(n-1)} \end{pmatrix} + \det\begin{pmatrix} y_1 & \cdots & & y_n \\ y_1'' & \cdots & & y_n'' \\ y_1'' & \cdots & & y_n'' \\ \vdots & & & \\ y_1^{(n-1)} & \cdots & & y_n^{(n-1)} \end{pmatrix} +$$

1st row (only) differentiated

2nd row only differentiated

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$$+ \cdots + \det \begin{pmatrix} y_1 & \cdots & y_n \\ \vdots & & \\ y_1^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{pmatrix}$$

last row differentiated only

Now every determinant on the right-hand side has a repeated row *except* the last one, this one.

So

$$W' = \det \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{pmatrix}$$

first $n-1$ rows like $W$, last row $n$th derivatives instead of $(n-1)$st deriv.

Now $\displaystyle y_j^{(n)}(x) = -\sum_{j=0}^{n-1} p_{n-j}\, y^{(j)}$ where we set $y^{(0)} = y$.

So

$$W' = -\sum_{j=0}^{n-1} \det \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \\ y_1^{(j)} & \cdots & y_n^{(j)} \end{pmatrix} \cdot p_{n-j}(x)$$

$= -p_1 W$ since only when $j = n-1$ is the determinant without a repeated row!

Examples

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

$$y_1 = e^{2x} \qquad y_2 = e^{3x}$$

$$W(y_1, y_2) = \det\begin{pmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{pmatrix} = 3e^{5x} - 2e^{5x} = e^{5x}$$

$$W' = 5e^{5x} = 5W = -(\underset{P_1}{-5})W$$

$$\frac{d^2y}{dx^2} + y = 0 \qquad y_1 = \sin x \qquad y_2 = \cos x$$

$$W = \det\begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} = -\sin^2 x - \cos^2 x = -1$$

$$W' = 0 = \underset{P_1}{0} \cdot W$$

$P_1$ (since $y'$ term is missing)